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MULTIPLE RECURRENCE AND ALMOST SURE CONVERGENCE FOR WEAKLY MIXING DYNAMICAL SYSTEMS

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ABSTRACT

We prove the following:

Let $(X, \mathcal{B}, \mu, T)$ be a weakly mixing dynamical system such that the restriction of T to its Pinsker algebra has singular spectrum, then for all positive integers H, for all $f_i \in L^\infty$, $1 \leq i \leq H$ the averages

$$\frac{1}{N}\sum_{n=1}^{N} f_1(T^n x) f_2(T^{2n}x)\cdots f_H(T^{Hn}x) \text{ converge a.e. to } \prod_{i=1}^{H}\int f_i d\mu.$$

1. Introduction

Let $(X, \mathcal{B}, \mu, T)$ be a weakly mixing dynamical system. The study of the averages

$$\frac{1}{N}\sum_{n=1}^{N} T^n f_1 \cdot T^{2n} f_2 \cdots T^{Hn} f_H, \quad f_1, f_2, \ldots, f_H \in L^\infty(\mu)$$

has been introduced by H. Furstenberg [Fu] in studying questions of multiple recurrence. Jointly with Y. Katznelson and D. Ornstein he proved the following result.

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THEOREM 1 ([FKO]): *Let $(X, \mathcal{B}, \mu, T)$ be a weakly mixing dynamical system. For all positive integers H, for all $f_1, f_2, \ldots, f_H \in L^\infty$ the averages*

$$\frac{1}{N}\sum_{n=1}^{N} T^n f_1 \cdot T^{2n} f_2 \ldots T^{Hn} f_H \text{ converge in } L^2 \text{ norm to } \prod_{i=1}^{H} f_i d\mu.$$

We refer to [Fu], for the original context in which the term multiple recurrence was used. In the same book he also raised the question of the a.e. convergence of these averages.

In [B], J. Bourgain treated the case $H = 2$ and proved that if T is an ergodic dynamical system and T_1 and T_2 are powers of T, then the averages $\frac{1}{N}\sum_{n=1}^{N} T_1^n f_1(x)\, T_2^n f_2(x)$ converge a.e. when N tends to infinity.

Our purpose is to prove the following theorem.

THEOREM 2: *Let $(X, \mathcal{B}, \mu, T)$ be a weakly mixing dynamical system such that the restriction of T to its Pinsker algebra has singular spectrum, then for all positive integers H, for all $f_i \in L^\infty$, $1 \le i \le H$, the averages*

$$\frac{1}{N}\sum_{n=1}^{N} f_1(T^n x) \cdot f_2(T^{2n} x) \cdots f_H(T^{Hn} x) \text{ converge a.e. to } \prod_{i=1}^{H}\int f_i d\mu.$$

Theorem 2 considerably reduces the possibility of finding a counterexample to H. Furstenberg's question for weakly mixing systems. It covers a very large class of weakly mixing systems. In the case of $H = 3$ we obtain a more precise result.

THEOREM 3: *Let $(X, \mathcal{B}, \mu, T)$ be a weakly mixing system. Consider $f \in L^2(\mu)$ and $\mathcal{P}$ the Pinsker algebra of T. Let $f^\infty = \mathbb{E}(f|\mathcal{P})$ and denote by $P(f^\infty)$ the projection of f^∞ (in $L^2(X, \mathcal{P}, \mu)$) onto the vector space of those functions whose spectral measure is absolutely continuous with respect to Lebesgue measure. Then for all $f_2, f_3 \in L^\infty(X, \mathcal{F}, \mu)$,*

(1)

$$\lim_N \frac{1}{N}\sum_{n=1}^{N} f(T^n x) f_2(T^{2n} x) f_3(T^{3n} x) - \frac{1}{N}\sum_{n=1}^{N} f^\infty(T^n x) f_2^\infty(T^{2n} x) f_3^\infty(T^{3n} x)$$
$$= 0 \text{ a.e.}$$

(2) *If we denote by $\tilde{f}_1$ the function $f^\infty - Pf^\infty$*

$$\frac{1}{N}\sum_{n=1}^{N} \tilde{f}_1(T^n x) \cdot f_2^\infty(T^{2n} x) \cdot f_3^\infty(T^{3n} x) \text{ converges a.e. to } \left(\int f d\mu\right)\left(\prod_{i=1}^{2}\int f_i d\mu\right)$$

for any $f_2, f_3 \in L^\infty(X, \mathcal{F}, \mu)$.

The proof of Theorem 2 follows the following lines.

(1) We reduce the convergence to the Pinsker algebra (maximum sub σ field on which T has zero entropy). (Proposition 4)

(2) We prove a result on the structure of pairwise independent joining (Theorem 5) that we develop from the proofs in [H]. This theorem and its corollary are of independent interest. In Section B we will give a contribution to the mixing of order 3 problem. In [A] we also use these results, among others, to prove a multiterm return times theorem for weakly mixing systems.

(3) We combine these theorems and a couple of arguments to derive first Theorem 3, then Theorem 2.

2. The results and proofs

(A) Reduction to the Pinsker algebra. The following proposition was proved by E. Lesigne [Le] in the case $H = 2$.

Proposition 4: *Let* $(X, \mathcal{F}, \mu, T)$ *be an ergodic dynamical system and* $\mathcal{P}$ *its Pinsker algebra. If we denote by* f^∞ *the projection of* f *onto* $L^2(\mathcal{P})$ *we have for all positive integers* H *for all* $f_i \in L^\infty(\mu), 1 \le i \le H$

$$\lim_N \left| \frac{1}{N} \sum_{n=1}^{N} f_1(T^n x) f_2(T^{2n} x) \cdots f_H(T^{Hn} x) - \frac{1}{N} \sum_{n=1}^{N} f_1^\infty(T^n x) f_2^\infty(T^{2n} x) \cdots f_H^\infty(T^{Hn} x) \right| = 0 \text{ a.e.}$$

Proof: First we recall a few points regarding the Pinsker algebra. The result will follow for each $T^i(\mathcal{A})$ and then, by density, for all L^∞ functions.

(a) There exists a σ algebra $\mathcal{A}$ such that

$$T^{-1}\mathcal{A} \subset \mathcal{A}; \quad \bigcap_n T^{-n}\mathcal{A} = \mathcal{P}; \quad \bigcup_n T^n \mathcal{A} \text{ is dense in } \mathcal{F}.$$

(b) If we denote by $\mathbb{E}(f|T^{-k}\mathcal{A})$ the conditional expectation of an L^1 function with respect to the σ field $T^{-k}(\mathcal{A})$, we have by the martingale convergence theorem

$$\lim_{k\to\infty} \mathbb{E}(f|T^{-k}\mathcal{A}) = \mathbb{E}(f|\mathcal{P}) = f^\infty \text{ a.e.}$$

(c) $\mathbb{E}(g \circ T^m | T^{-n}\mathcal{A}) = \mathbb{E}(g|T^{-(n-m)}\mathcal{A}) \circ T^m = f^\infty$.

We can assume that $\|f_\ell\|_\infty \leq 1$ for $1 \leq \ell \leq H$ and the f_ℓ are $\mathcal{A}$ measurable. For simplicity of notation we will prove the case $H = 3$. The reader will see that the method extends without difficulty to any positive integer H. The path is the following.

We show that

(i)
$$\lim_N \left(\frac{1}{N} \sum_{n=1}^{N} f_1(T^n x) \cdot f_2(T^{2n}x) f_3(T^{3n}x) - \frac{1}{N} \sum_{n=1}^{N} f_1^\infty(T^n x) f_2(T^{2n}x) f_3(T^{3n}x) \right) = 0 \text{ a.e.}$$

(ii)
$$\lim_N \left(\frac{1}{N} \sum_{n=1}^{N} f_1^\infty(T^n x) f_2(T^{2n}x) f_3(T^{3n}x) - \frac{1}{N} \sum_{n=1}^{N} f_1^\infty(T^n x) f_2^\infty(T^{2n}x) f_3(T^{3n}x) \right) = 0 \text{ a.e.}$$

(iii)
$$\lim_N \left(\frac{1}{N} \sum_{n=1}^{N} f_1^\infty(T^n x) f_2^\infty(T^{2n}x) f_3(T^{3n}x) - \frac{1}{N} \sum_{n=1}^{N} f_1^\infty(T^n x) f_2^\infty(T^{2n}x) f_3^\infty(T^{3n}x) \right) = 0 \text{ a.e.}$$

Combining (i), (ii), and (iii) we obtain a proof of the proposition.

Step (i): We fix $\epsilon > 0$ and consider a positive integer k such that

$$\text{(i}_0\text{)} \qquad \|E(f_1 | T^{-k}\mathcal{A}) - f_1^\infty\| < \epsilon.$$

We consider for r, where $0 \leq r \leq k-1$,

(i$_1$)
$$\frac{1}{N} \sum_{n=1}^{N} (f_1 - \mathbb{E}(f_1)(T^{-k}\mathcal{A}))(T^{nk+r}x) f_2(T^{2(nk+r)}x) f_3(T^{3(nk+r)}x) = \frac{1}{N} \sum_{n=1}^{N} H_n.$$

For k and r we compute

$$\text{(i}_2\text{)} \quad \mathbb{E}[(f_1 - \mathbb{E}(f_1 | (T^{-k}\mathcal{A})) \circ T^{nk+r} \cdot_2 \circ T^{2(nk+r)} \cdot f_3 \circ T^{3(nk+r)} | T^{-[(n+1)k+r]}\mathcal{A}] = \mathbb{E}(H_n | T^{-((n+1)k+r)}\mathcal{A}).$$

The functions $f_2 \circ T^{2(nk+r)}$ and $f_3 \circ T^{3(nk+r)}$ are $T^{-((n+1)k+r)}\mathcal{A}$ measurable for $n \geq 2$. Thus (i_2) is equal to

$$f_2 \circ T^{2(nk+r)} \cdot f_3 \circ T^{3(nk+r)} \cdot \mathbb{E}[(f_1 - \mathbb{E}(f_1|T^{-k}\mathcal{A})) \circ T^{nk+r}|T^{-[(n+1)k+r]}\mathcal{A}]$$

which, by (c), is equal to zero.

We can conclude that the averages in (i_1) converge a.e. to zero. We have

$$\sum_{N=1}^{\infty} \int \left| \frac{1}{N^2} \sum_{n=1}^{N^2} H_n \right|^2 d\mu < \infty$$

as $\int H_i H_j d\mu = 0$ for $i \neq j$. Then using the boundedness of H_n we obtain

$$\lim_N \frac{1}{N} \sum_{n=1}^{N} H_n = 0 \text{ a.e.}$$

This being true for each r, $0 \leq r \leq k-1$ we conclude also that the averages

$$(\mathrm{i}_3) \qquad \frac{1}{N} \sum_{n=1}^{N} (f_1 - \mathbb{E}(f_1|T^{-k}\mathcal{A}))(T^n x) f_2(T^{2n}x) f_3(T^{3n}x)$$

converge a.e. to zero. To reach the conclusion in Step (i) it is enough to evaluate

$$\frac{1}{N} \sum_{n=1}^{N} [\mathbb{E}(f_1|T^{-k}\mathcal{A}) - f_1^{\infty}](T^n x) f_2(T^{2n}x) f_3(T^{3n}x).$$

By Birkhoff's theorem we have

$$\begin{aligned}
&\overline{\lim_N} \left| \frac{1}{N} \sum_{n=1}^{N} (\mathbb{E}(f_1|T^{-k}\mathcal{A}) - f_1^{\infty})(T^n x) f_2(T^{2n}x) f_3(T^{3n}x) \right| \\
&\leq \overline{\lim_N} \frac{1}{N} \sum_{n=1}^{N} |\mathbb{E}(f_1|T^{-k}\mathcal{A}) - f_1^{\infty}|(T^n x) \qquad (\text{as } \|f_\ell\|_\infty \leq 1 \text{ for } \ell = 2,3) \\
&= \int |\mathbb{E}(f_1|T^{-k}\mathcal{A}) - f_1^{\infty}| d\mu < \epsilon \qquad \text{by } (\mathrm{i}_0)
\end{aligned}$$

(as T is eigodic). As ϵ is arbitrary we have shown that

$$\frac{1}{N} \sum_{n=1}^{N} (f_1 - f_1^{\infty}(T^n x)) f_2(T^{2n}x) \cdot f_3(T^{3n}x) \qquad \text{converge a.e. to } 0.$$

STEP (ii): The proof is analogous to Step (i). We only sketch it. Again we start with $\epsilon > 0$ and fix k such that

$$\text{(ii)}_0 \qquad \|E(f_2|T^{-2k}\mathcal{A}) - f_2^\infty\| < \epsilon.$$

Then for $0 \le r \le k-1$ we consider

$$\text{(ii)}_1 \qquad \frac{1}{N}\sum_{n=1}^{N} f_1^\infty(T^{nk+r}x)(f_2 - \mathbb{E}(f_2|T^{-2k}\mathcal{A}))(T^{2(nk+r)}x)\cdot f_3(T^{3(nk+r)}x).$$

The functions $(f_1^\infty \circ T^{nk+r})\cdot((f_2 - \mathbb{E}(f_2|T^{-2k}\mathcal{A}))\circ T^{2(nk+r)})\cdot f_3\circ T^{3(nk+r)}$ are $T^{-(2(nk+r))}\mathcal{A}$ measurable, as $f_1^\infty \circ T^{nk+r}$ is $\mathcal{P}$ measurable. We compute the conditional expectation of

$$f_1^\infty \circ T^{nk+r}\cdot(f_2 - \mathbb{E}(f_2|T^{-2k}\mathcal{A}))\circ T^{2(nk+r)}\cdot f_3\circ T^{3(nk+r)}$$

with respect to $T^{-2[(n+1)k+r]}\mathcal{A}$ for $n \ge 2$.

As previously, we obtain 0 and conclude that the averages in (ii)_1 converge a.e. to zero. Thus the averages

$$\frac{1}{N}\sum_{n=1}^{N} f_1^\infty(T^n x)(f_2 - \mathbb{E}(f_2|T^{-2k}\mathcal{A})(T^{2n}x)f_3(T^{3n}x) \qquad \text{converge a.e. to 0.}$$

Finally we evaluate the averages

$$\frac{1}{N}\sum_{n=1}^{N} f_1^\infty(T^n x)(\mathbb{E}(f_2|(T^{-2k}\mathcal{A}) - f_2^\infty)(T^{2n}x)f_3(T^{3n}x).$$

Using (ii)_0, $\|f_1^\infty\|_\infty \le 1$ as $f_1^\infty = \mathbb{E}(f|\mathcal{P})$, $\|f_3^\infty\|_\infty \le 1$, we obtain

$$\begin{aligned}\int \overline{\lim_N}\left|\frac{1}{N}\sum_{n=1}^{N} f_1^\infty(T^n x)(\mathbb{E}(f_2|T^{-2k}\mathcal{A}) - f_2^\infty)(T^{2n}x)f_3(T^{3n}x)\right|d\mu \\ \le \int \overline{\lim_N}\frac{1}{N}\sum_{n=1}^{N}(|\mathbb{E}(f_2|T^{-2k}\mathcal{A}) - f_2^\infty|(T^n x)d\mu \\ = \int |\mathbb{E}(f_2|T^{-2k}\mathcal{A}) - f_2^\infty|d\mu < \epsilon\end{aligned}$$

by Birkhoff's theorem applied to T^2. These steps give us

$$\lim_N\left|\frac{1}{N}\sum_{n=1}^{N} f_1^\infty(T^n x)f_2(T^{2n}x)f_3(T^{3n}x) - \frac{1}{N}\sum_{n=1}^{N} f_1^\infty(T^n x)f_2^\infty(T^{2n}x)f_3(T^{3n}x)\right| = 0 \text{ a.e.}$$

STEP (iii): We believe the process is now clear. The starting point is again $\epsilon > 0$, then k fixed, so that

$$\|\mathbb{E}(f_3|T^{-3k}\mathcal{A}) - f_3^\infty\| < \epsilon.$$

Then for $0 \leq r \leq k-1$ we consider

$$\frac{1}{N}\sum_{n=1}^{N} f_1^\infty(T^{nk+r}x) f_2^\infty(T^{2(nk+r)}x)(f_3 - \mathbb{E}(f_3)T^{-3k}\mathcal{A})(T^{3(nk+r)}x).$$

We obtain a martingale by considering the sigma fields $(T^{-3(nk+r)}\mathcal{A})_{n\geq 3}$. We conclude by using the limsup and Birkhoff's theorem.

(B) ON THE STRUCTURE OF PAIRWISE INDEPENDENT JOINING.

Definitions: Given an integer $r \geq 2$ and a joining of r dynamical systems $(X_i, \mathcal{B}_i, \mu_i, T_i)$, $1 \leq i \leq r$ is a probability measure w on the product space $\prod_i (X_i, \mathcal{B}_i)$ which is invariant under the diagonal transformation $\prod_i T_i$ and whose marginal projection on each X_i is equal to μ_i; the joining w is pairwise independent if its projection on $X_i \times X_j$ is equal to $\mu_i \times \mu_j$ for all $\neq j$, and it is independent if it is the product measure.

- A system $(X, \mathcal{B}, \mu, T)$ is mixing of order 3 if, for every $A_i \in \mathcal{B}$ $(1 \leq i \leq 3)$,

$$\mu(A_1 \cap T^{n_2}A_2 \cap T^{n_3}A_3) \text{ converges to } \prod_{i=1}^{3}\mu(A_i) \text{ when } n_2 \to \infty, n_3 - n_2 \to \infty.$$

- A function $f_1 \in L^2(\mu)$ generates the mixing of order 3 property for the system $(X, \mathcal{B}, \mu, T)$ if for all functions $f_2, f_3 \in L^\infty(\mu)$

$$\int f_1(x) f_2(T^{n_2}x) \cdot f_3(T^{n_3}x) d\mu \text{ converge to } \prod_{i=1}^{3}\int f_i d\mu.$$

- From now on we can and will assume that the dynamical systems are standard, i.e., X is compact, $\mathcal{B}(X)$ = the set of Borel sets of X, and T is a homeomorphism on X.

In this section we will prove the following results.

THEOREM 5: *Let $(X_1, \mathcal{B}_1, \mu_1, T_1)$ be a weakly mixing system. Take $f \in L^2(\mu_1)$ and denote by Pf the projection of f onto the vector space of those functions whose spectral measure is absolutely continuous with respect to Lebesgue measure m.*

Let w be a pairwise independent joining of $(X_1, \mathcal{B}_1, \mu_1, T_1)$ with two ergodic systems $(X_2, \mathcal{B}_2, \mu_2, T_2)$ and $(X_3, \mathcal{B}_3, \mu_3, T_3)$, one of them being weakly mixing. Then for all $f_2, f_3 \in L^\infty$ we have

$$\int f(x_1)f_2(x_2)f_3(x_3)dw(x_1,x_2,x_3) = \left(\int f d\mu_1\right)\left(\int f_2 d\mu_2\right)\left(\int f_3 d\mu_3\right) + \int Pf(x_1)f_2(x_2)\cdot f_3(x_3)dw(x_1,x_2,x_3).$$

The corollary is then immediate, by taking $Pf = 0$.

COROLLARY: *Let $(X_1, \mathcal{B}_1, \mu_1, T_1)$ be a weakly mixing system and $f_1 \in L^2(\mu_1)$ such that $\sigma_{f_1} \perp m$ (where σ_{f_1} is the spectral measure of f_1 and m is the Lebesgue measure). Then for all pairwise independent joinings w of $(X_1, \mathcal{B}_1, \mu_1, T_1)$ with two ergodic systems $(X_2, \mathcal{B}_2, \mu_2, T_2)$ and $(X_3, \mathcal{B}_3, \mu_3, T_3)$, one of them being weakly mixing, we have*

$$\int f_1(x_1)f_2(x_2)f_3(x_3)dw = \left(\int f_1 d\mu_1\right)\left(\int f_2 d\mu_2\right)\left(\int f_3 d\mu_3\right).$$

THEOREM 6: *Let $(X, \mathcal{B}, \mu, T)$ be a mixing system and $f \in L^2(\mu)$ such that $\sigma_f \perp m$. Then f generates the mixing of order 3 property.*

Proof of Theorem 5: Our proof mainly follows the lines of [H]. The main point is to show that some of the arguments presented in [H] can be made "local".

LEMMA 1: *Let $(X_1, \mathcal{B}_1, \mu_1, T_1)$ be a weakly mixing system. Then any $f \in L^2(\mu)$ can be written as $f = f_1 + f_2$ where $\sigma_{f_2} \ll m$, $\int f_1 d\mu = \int f d\mu$ and $f_1 - \int f d\mu = \bar{f}_1$ is such that $\sigma_{\bar{f}_1} \perp m$ and $\sigma_{\bar{f}_1}$ is continuous.*

Proof of Lemma 1: It is a simple consequence of the spectral theorem.

We can decompose $L^2(\mu_1) = H_1 \oplus H_2$, where

$$H_2 = \{f \in L^2(\mu_1) : \sigma_f \ll m\} \quad \text{and} \quad H_2 = \{f \in L^2(\mu_1) : \sigma_f \perp m\}.$$

Then $f = f_1 + f_2$ with $f_1 \in H_1$ and $f_2 \in H_2$. As σ_{f_2} is continuous we have $\int f_2 d\mu_1 = 0$. Thus $\int f_1 d\mu_1 = \int f d\mu$. The transformation T_1 being weakly mixing, we have

$$f_1 - \int f_1 d\mu_1 = f_1 - \int f d\mu = \bar{f}_1,$$

and this function $\bar{f}_1$ has continuous spectral measure $\sigma_{\bar{f}_1}$.

Finally, as $\mathbb{C} \subset H$, we have $\sigma_{\bar{f}_1} \perp m$.

End of the proof of Theorem 5: By Theorem 3 [H], for w a pairwise independent joining of $(X_i, \mathcal{B}_i, \mu_i, T_i), 1 \leq i \leq 3$ where $(X_1, \mathcal{B}_1, \mu_1, T_1)$ satisfies the assumption of Lemma 1, $g_2, g_3 \in L^\infty(\mu_i)$ (for $i = 2, 3$). We know that there exists a finite complex measure τ on $\mathbb{T}^3$,

$$\hat{\tau}(m,n,p) = \int \bar{f}_1(T_1^m x_1) g_2(T_2^n x_2) g_3(T_3^p x_3) dw(x_1, x_2, x_3).$$

This measure τ is concentrated on the closed subgroup

$$H = \{(x_1, x_2, x_3) \in \mathbb{T}^3 : x_1 + x_2 + x_3 = 0\}.$$

Furthermore, the images of $|\tau|$ by the natural projections $(\pi_{1,2}, \pi_{1,3}, \pi_{2,3})$ of $\mathbb{T}^3$ on $\mathbb{T}^2$ are absolutely continuous with respect to $\sigma_{f_1} \times \sigma_{f_2}, \sigma_{\bar{f}_1} \times \sigma_{g_3}$ and $\sigma_{g_2} \times \sigma_{g_3}$, respectively.

The goal is to show that $\hat{\tau}(0) = 0$, which will follow from $\tau = 0$.

By Theorem 5 [H], this measure has the following property: each projection of $\rho = |\tau|$ on $\mathbb{T}$ is the sum of a discrete measure and an absolutely continuous measure (with respect to Lebesgue measure). In particular the first projection ρ_1 of ρ on $\mathbb{T}$ is the sum of some discrete measure and some absolutely continuous measure with respect to m (Lebesgue measure).

But ρ_1 is $\ll \sigma_{\bar{f}_1}$ which is $\perp m$, by Lemma 1. Thus $\rho_1 = 0 \Rightarrow \rho = 0$ and $\tau = 0$. In particular $\hat{\tau}(0) = 0$, which means that

$$\int \bar{f}_1(x_1) g_2(x_2) g_3(x_3) dw(x_1, x_2, x_3) = 0.$$

Thus

$$\int f_1(x_1) g_2(x_2) g_3(x_3) dw(x_1, x_2, x_3) = \left(\int f_1 d\mu_1\right) \int g_2(x_2) g_3(x_3) dw$$
$$= \left(\int f d\mu\right) \cdot \left(\int g_2 d\mu_2\right) \left(\int g_3 d\mu_3)\right)$$

as the projection of w on (2,3) is the product measure $\mu_2 \times \mu_3$. ∎

Proof of the Corollary: The proof is immediate. We simply take $Pf = 0$.

Proof of Theorem 6: We consider a mixing dynamical system. As we said at the beginning of section B, we can assume that $(X, \mathcal{B}, \mu, T)$ is a "standard" dynamical system.

We take $f \in L^2(\mu)$ with $\sigma_f \perp m$. We need to show that for all $f_2, f_3 \in L^\infty(\mu)$ we have

$$(**) \qquad \lim_{\substack{n_1 \to \infty \\ n_3 - n_2 \to \infty}} \int f(x) f_2(T^{n_2}x) f_3(T^{n_3}x) d\mu = \left(\int f d\mu\right) \cdot \left(\prod_{i=2}^{3} \int f_i d\mu\right).$$

If we assume $(**)$ false then we could find $f_2, f_3 \in L^\infty(\mu)$ such that

$$\lim_{k \to \infty} \int f(x) f_2(T^{n_2^k}x) \cdot f_3(T^{n_3^k}x) d\mu \neq \left(\int f d\mu\right)\left(\prod_{i=2}^{3} \int f_i d\mu\right).$$

By a diagonal process on the set of continuous function $\mathcal{C}(X)$, we can extract a subsequence $(n_2^{k'}, n_3^{k'})$ of (n_2^k, n_3^k) such that $\forall h_1, h_2, h_3 \in \mathcal{C}(X)$

$$\lim_{k' \to \infty} \int h_1(x) h_2(T^{n_2^{k'}} x) h_3(T^{n_3^{k'}} x) d\mu \text{ exists.}$$

This limit defines a measure w which is a pairwise independent joining by the mixing property $(X, \mathcal{B}, \mu, T)$.

By density of $\mathcal{C}(X)$ in $L^2(\mu)$ we also have

$$\lim_{k' \to \infty} \int f(x) f_2(T^{n_2^{k'}} x) f_3(T^{n_3^{k'}} x) d\mu = \int f(x_1) f_2(x_2) f_3(x_3) dw.$$

By the corollary to Theorem 5 we have

$$\int f(x_1) f_2(x_2) f_3(x_3) dw = \int f d\mu \int f_2 d\mu \int f_3 d\mu,$$

which is a contradiction. This proves Theorem 6.

(C) Proofs of Theorems 2 and 3.

(C$_1$) *Proof of Theorem 3:* Part (1) of this theorem is a direct consequence of Proposition 4. It is enough to consider the restriction of T to the Pinsker algebra $\mathcal{P}$. We assume that $(X, \mathcal{P}, \mu, T)$ is a standard dynamical system. We consider $\tilde{f}_1 \in L^2(X, \mathcal{P}, \mu)$ such that $\sigma_{\tilde{f}_1} \perp m$. We will reach a proof of part (2) of this theorem after several steps:

STEP 1: We can find a set of full measure $\underline{\tilde{\tilde{X}}}$ such that for all $x \in \underline{\tilde{\tilde{X}}}, \forall h, g \in \mathcal{C}(X)$ we have

$$\frac{1}{N}\sum_{n=1}^{N} h(T^n x) g(T^{2n}x) \text{ converge to } \int h d\mu \int g d\mu, \tag{1}$$

$$\frac{1}{N}\sum_{n=1}^{N} h(T^{2n} x) g(T^{3n}x) \text{ converge to } \int h d\mu \cdot \int g d\mu, \tag{2}$$

$$\frac{1}{N}\sum_{n=1}^{N} h(T^n x) g(T^{3n}x) \text{ converge to } \int h d\mu \cdot \int g d\mu. \tag{3}$$

This first step can be proved by using [B] (for the a.e.). This result guarantees the convergence of elements in (1), (2), and (3) for a countable dense set of continuous functions. The identification of the limit comes from the norm convergence result of [FKO].

STEP 2: Given $\tilde{f}_1 \in L^2(\mu)$ there exists a sequence h_i of continuous functions such that

$$\lim_i \sup_N \left[\frac{1}{N}\sum_{n=1}^{N} |\tilde{f}_1 - h_i|(T^n x)|f_2(T^{2n}x)||f_3(T^{3n}x)|\right] = 0$$

for all $x \in \underline{X}'$ set of full measure, $f_2, f_3 \in \mathcal{C}(X), |f_2| \leq 1, |f_3| \leq 1$.

To prove this we can use the maximal inequality and the density of $\mathcal{C}(X)$ in $L^2(\mu)$. We have

$$\sup_N \left[\frac{1}{N}\sum_{n=1}^{N} |\tilde{f}_1 - h_i|(T^n x)|f_2(T^{2n}x)|f_3(T^{3n}x)|\right] \leq \sup_N \left[\frac{1}{N}\sum_{n=1}^{N} |\tilde{f}_1 - h_i|(T^n x)\right].$$

By selecting h_i such that $\sum_{i=1}^{\infty} \|\tilde{f}_1 - h_i\|_1 < \infty$ we get for each $\lambda > 0$

$$\sum_{i=1}^{\infty} \mu\left\{x : \sup_N \left[\frac{1}{N}\sum_{n=1}^{N} |\tilde{f}_1 - h_i|(T^n x)\right] > \lambda\right\} \leq \frac{1}{\lambda}\sum_{i=1}^{\infty}\int |\tilde{f}_1 - h_i| d\mu < \infty.$$

A simple application of the Borel–Cantelli lemma gives the proof of Step 2.

STEP 3: We want to prove that for $x \in \underline{X}' \cap \underline{\tilde{\tilde{X}}}$,

$$\overline{\lim} \frac{1}{N}\sum_{n=1}^{N} \tilde{f}_1(T^n x) f_2(T^{2n}x) f_3(T^{3n}x) = \left(\int \tilde{f}_1 d\mu\right)\left(\int f_2 d\mu\right)\left(\int f_3 d\mu\right)$$

for all $f_2, f_3 \in \mathcal{C}(X)$.

First there exists a sequence $N_k(x)$ such that

$$\overline{\lim}\frac{1}{N}\sum_{n=1}^{N}\tilde{f}_1(T^n x)f_2(T^{2n}x)f_3(T^{3n}x)$$
$$= \lim_k \frac{1}{N_k(x)}\sum_{n=1}^{N_k(x)}\tilde{f}_1(T^n x)f_2(T^{2n}x)f_3(T^{3n}x).$$

By a diagonal process we can extract a subsequence $M_k(x)$ such that

$$\lim_k \frac{1}{M_k(x)}\sum_{n=1}^{M_k(x)} G(T^n x, T^{2n}x, T^{3n}x) = \int G(x_1,x_2,x_3)dw(x_1,x_2,x_3)$$

for all $G \in \mathcal{C}(X \times X \times X)$. The measure w is a joining which is also pairwise independent by Step 1.

By Step 2 we have

$$\lim_k \frac{1}{M_k(x)}\sum_{n=1}^{M_k(x)}\tilde{f}_1(T^n x)f_2(T^{2n}x)f_3(T^{3n}x) = \lim_{j\to\infty}\int h_j(x_1)f_2(x_2)f_3(x_3)dw.$$

As $\|h_j - f\| \underset{j}{\to} 0$ we have

$$\lim_{j\to\infty}\int h_j(x_1)f_2(x_2)f_3(x_3)dw = \int \tilde{f}_1(x_1)f_2(x_2)f_3(x_3)dw.$$

So

$$\lim_{k\to\infty}\frac{1}{M_k(x)}\sum_{n=1}^{M_k(x)}\tilde{f}_1(T^n x)f_2(T^{2n}x)f_3(T^{3n}x) = \int \tilde{f}_1(x_1)f_2(x_2)f_3(x_3)dw.$$

As $\sigma_{\tilde{f}_1} \perp m$ we have by the corollary to Theorem 5

$$\int \tilde{f}_1(x_1)f_2(x_2)f_3(x_3)dw = \left(\int \tilde{f}_1 d\mu\right)\left(\int f_2 d\mu\right)\left(f_3 d\mu\right)$$
$$= \left(\int f d\mu\right)\left(\int f_2 d\mu\right)\left(\int f_3 d\mu\right)$$

(as $\int f d\mu = \int \tilde{f}_1 dw$).

By using similar arguments for

$$\underline{\lim}\frac{1}{N}\sum_{n=1}^{N}\tilde{f}_1(T^n x)f_2(T^{2n}x)f_3(T^{3n}x)$$

we obtain a proof of Theorem 3.

(C_2) *Proof of Theorem 2:* The ideas of the proof are similar to the ones used in Theorem 3. First, by Proposition 4 we can reduce the proof to the Pinsker algebra of T.

The system $(X,\mathcal{P},\mu,T)$ is weakly mixing and has singular spectrum. We can assume that it is "standard" to be able to deal with continuous functions. For the continuous functions $f_1^\infty, f_2^\infty,\dots,f_H^\infty$ we construct w a pairwise independent joining for x in a subset of X of full measure by considering

$$\overline{\lim_N}\frac{1}{N}\sum_{n=1}^{N}f_1^\infty(T^n x)f_2^\infty(T^{2n}x)\cdots f_H^\infty(T^{Hn}x)$$

and using the a.e. convergence of $\frac{1}{N}\sum_{n=1}^{N} g_1^\infty(T^{in}x)gj^\infty(T^{jn}x)$ to $(\int g_i^\infty d\mu)(\int g_j^\infty d\mu)$, $i\neq j$ for all $g_i^\infty, g_j^\infty\in\mathcal{C}(X)$.

As $(X,\mathcal{P},\mu,T^P), p\in\mathbb{N}^*$ is also weakly mixing with purely singular spectrum we can use Theorem 2 in [H] to conclude that this joining w is independent. This shows that

$$\overline{\lim_N}\frac{1}{N}\sum_{n=1}^{N}f_1^\infty(T^n x)f_2^\infty(T^{2n}x)\cdots f_H^\infty(T^{nH}x)=\prod_{i=1}^{H}\int f_i^\infty d\mu.$$

A similar argument for $\underline{\lim}$ gives the a.e. convergence of the averages $\frac{1}{N}\sum_{n=1}^{N} f_1^\infty\ (T^n x)f_2^\infty(T^{2n}x)\cdots f_H^\infty(T^{nH}x)$ to $\prod_{i=1}^{\infty}\int f_i^\infty d\mu$. Combining Proposition 4 and the last conclusion we obtain a proof of Theorem 2.

Remark: As mentioned in the introduction, the methods used in this paper can be used to obtain a multiterm return times theorem for weakly mixing systems with some indication of the good universal set. This is done in [A].

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